

Twisting all the way: from algebras to morphisms and connections¹

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Abstract

Given a Hopf algebra H and an algebra A that is an H -module algebra we consider the category of left H -modules and A -bimodules ${}^H_A\mathcal{M}_A$, where morphisms are just right A -linear maps (not necessarily H -equivariant). Given a twist \mathcal{F} of H we then quantize (deform) H to $H^\mathcal{F}$, A to A_\star and correspondingly the category ${}^H_A\mathcal{M}_A$ to ${}^{H^\mathcal{F}}_{A_\star}\mathcal{M}_{A_\star}$. If we consider a quasitriangular Hopf algebra H , a quasi-commutative algebra A and quasi-commutative A -bimodules, we can further construct and study tensor products over A of modules and of morphisms, and their twist quantization.

This study leads to the definition of arbitrary (i.e., not necessarily H -equivariant) connections on quasi-commutative A -bimodules, to extend these connections to tensor product modules and to quantize them to A_\star -bimodule connections. Their curvatures and those on tensor product modules are also determined.

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¹Paper based on joint work with Alexander Schenkel

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1 Introduction

Consider the basic algebraic structures underlying the differential geometry of a manifold M : the algebra $A = C^\infty(M)$ of complex valued functions on M ; the A -module of sections of the tangent bundle (i.e., vector fields), and that of one-forms; the algebra of tensor fields (\mathcal{T}, \otimes_A) and the exterior algebra (Ω^\bullet, \wedge) . Typical maps between these algebraic structures are the exterior derivative, connections, and tensor fields (like a metric tensor, a curvature tensor, etc.). The Lie algebra of vector fields (infinitesimal local diffeomorphisms of M) acts on all the above structures. The universal enveloping algebra of the Lie algebra of vector fields is a Hopf algebra H and it also acts on all the above structures.

In [1, 2] we have deformed the Hopf algebra H of vector fields on a manifold M via a Drinfeld twist (or twisting element) $\mathcal{F} \in H \otimes H$ [3]. Since the A -modules of vector fields and of one-forms, and the tensor and exterior algebras (\mathcal{T}, \otimes_A) and (Ω^\bullet, \wedge) carry a representation of the algebra H we have been able to deform these algebraic structures as well. Concerning morphisms, in [1, 2] we study in particular the deformations of the Lie derivative and inner derivative. Since H is the Hopf algebra associated to the Lie algebra of vector fields, interesting morphisms are not in general H -equivariant (they indeed transform covariantly). This deformed geometry has then been used to formulate a noncommutative gravity theory. Notice that the noncommutative connection and metric tensor considered in [1, 2] cannot be H -equivariant because they are dynamical fields.

In the present paper, outlining joint work with Alexander Schenkel [4], we clarify and further study the twist quantization or twist deformation scheme. Even if our leading example is the twist deformation of the algebraic structures of a manifold M , we here more in general consider a Hopf algebra H and A -modules where H is not necessarily cocommutative like the universal enveloping algebra of a Lie algebra (for example it can be a quantum group) and A is not necessarily commutative.

More precisely in this paper we study the category of A -modules carrying an action of a Hopf algebra H . The obvious morphisms in this category are A -module morphisms that are also H -equivariant (H -module morphisms). As we have seen, this choice is too restrictive and we therefore study the category ${}^H\mathcal{M}_A$ of right A -modules that carry also an H -action, but where morphisms are just right A -linear maps. We will also study just linear (not A -linear) maps, this is propaedeutical and it is also needed in order to understand deformations of connections. We further study the category ${}^H_A\mathcal{M}_A$ of left and right A -modules (A -bimodules) so that we can consider the tensor product over A of A -bimodules (the algebraic analogue of tensor product of vector bundles). In the category ${}^H_A\mathcal{M}_A$ morphisms are just right A -module morphisms; their tensor product will be defined for a subclass of noncommutative algebras A and A -bimodules: quasi-commutative algebras and bimodules carrying an action of a quasitriangular Hopf algebra H . Examples of such modules with a triangular Hopf algebra action naturally arise when considering twist deformations of commutative algebras and cocommutative Hopf algebras.

All these structures can be deformed via a Drinfeld twist $\mathcal{F} \in H \otimes H$ that in particular acts nontrivially on morphisms. If the algebra A is commutative then the twist deformed algebra A_\star is usually noncommutative, it is a quantization of A . We call twist quantization or simply quantization the twist deformation of the algebra A to A_\star , and of the category ${}^H_A\mathcal{M}_A$ to ${}^{H^\mathcal{F}}_{A_\star}\mathcal{M}_{A_\star}$, also when A is not commutative.

Having understood the properties of linear and A -linear morphisms we can then present our theory of connections. These are just right A -module connections. Quasitriangularity of the Hopf algebra H and quasi-commutativity of the A -bimodules then imply that these connections are also quasi-left A -module

connections. These connections are more general than the ones studied in the literature on A -bimodules [5, 6, 7]. In the restrictive case they are also H -equivariant then they match the definition in [5, 6, 7].

Given two connections on two quasi-commutative A -bimodules V and W we next define their sum as the connection on the tensor product module $V \otimes_A W$ (physically this is relevant for example when considering the covariant derivative of composite fields). Again this differs from the sum of bimodule connections discussed in the literature [8, 7, 9]. We further study the corresponding quantized connections as well as their curvatures. Twist deformation acts nontrivially on these structures, for example flat connections are twisted in non flat ones and vice versa.

2 Twisting algebras, modules and morphisms

2.1 Hopf algebras, modules, twists; twisted algebras and modules

We start settling the notation and recalling the structures we will later deform. In deformation quantization the field of complex numbers \mathbb{C} is replaced by the ring $\mathbb{C}[[\hbar]]$ of formal power series (in an indeterminate, say \hbar) with coefficients in \mathbb{C} . In order to cover also this example (that is rich of twists, an example being the Moyal-Weyl twist) we consider modules and algebras over a commutative ring \mathbb{K} with unit element $1 \in \mathbb{K}$. Then vector spaces over \mathbb{C} are replaced by \mathbb{K} -modules, and \mathbb{C} -linear maps are replaced by \mathbb{K} -linear maps (\mathbb{K} -module morphisms).

We recall that a Hopf algebra H is an algebra (over \mathbb{K}) with multiplication $\mu : H \otimes H \rightarrow H$ and two algebra morphisms $\Delta : H \rightarrow H \otimes H$ (coproduct), $\varepsilon : H \rightarrow \mathbb{K}$ (counit) and a \mathbb{K} -linear map $S : H \rightarrow H$ (antipode) satisfying, for all $\xi \in H$, $(\Delta \otimes \text{id})\Delta(\xi) = (\text{id} \otimes \Delta)\Delta(\xi)$ and

$$(\varepsilon \otimes \text{id})\Delta(\xi) = (\text{id} \otimes \varepsilon)\Delta(\xi) = \xi \quad , \quad \mu((S \otimes \text{id})\Delta(\xi)) = \mu((\text{id} \otimes S)\Delta(\xi)) = \varepsilon(\xi)1 \quad , \quad (2.1)$$

where 1 is the unit in H . As usual we use Sweedler's notation for the coproduct, for all $\xi \in H$, $\Delta(\xi) = \xi_1 \otimes \xi_2$ (sum understood), similarly $(\Delta \otimes \text{id})\Delta(\xi) = (\text{id} \otimes \Delta)\Delta(\xi) = \xi_1 \otimes \xi_2 \otimes \xi_3$.

A left module V over an algebra A (or **left A -module**) is a \mathbb{K} -module V with a \mathbb{K} -linear map $\cdot : A \otimes V \rightarrow V$ satisfying, for all $a, a' \in A$ and $v \in V$,

$$(a a') \cdot v = a \cdot (a' \cdot v) \quad , \quad 1 \cdot v = v \quad . \quad (2.2)$$

The map $\cdot : A \otimes V \rightarrow V$ is called an action of A on V or a representation of A on V . We denote the category of left A -modules by ${}_A\mathcal{M}$. Morphisms in this category are left A -linear maps. Similarly \mathcal{M}_A denotes the category of right A -modules. A left A -module and a right A -module structure on V are compatible if left and right A -actions commute, for all $a, a' \in A$, and $v \in V$, $(a \cdot v) \cdot a' = a \cdot (v \cdot a')$. Then V is an A -bimodule. We denote the corresponding category by ${}_A\mathcal{M}_A$.

A **left H -module algebra** is an algebra A which is also a left H -module (where the left action is denoted by \triangleright), such that for all $\xi \in H$ and $a, b \in A$,

$$\xi \triangleright (ab) = (\xi_1 \triangleright a) (\xi_2 \triangleright b) \quad , \quad \xi \triangleright 1 = \varepsilon(\xi)1 \quad . \quad (2.3)$$

We define ${}^H\mathcal{A}$ to be the category of H -module algebras where morphisms are algebra morphisms $\rho : A \rightarrow \tilde{A}$ that are *not* necessarily also H -module morphisms (i.e. H -equivariant): for all $\xi \in H$, $a \in A$, $\rho(\xi \triangleright a) = \xi \tilde{\triangleright} \rho(a)$.

We can now consider A -bimodules V , where $A \in {}^H\mathcal{A}$ and V is also a left H -module. Compatibility between the Hopf algebra structure of H and the A -bimodule structure of V leads to the following covariance requirement:

Definition 2.1. A **left H -module A -bimodule** V is an A -bimodule V over $A \in {}^H\mathcal{A}$ which is also a left H -module, such that for all $\xi \in H$, $a \in A$ and $v \in V$,

$$\xi \triangleright (a \cdot v) = (\xi_1 \triangleright a) \cdot (\xi_2 \triangleright v) \quad , \quad \xi \triangleright (v \cdot a) = (\xi_1 \triangleright v) \cdot (\xi_2 \triangleright a) \quad . \quad (2.4)$$

The category of left H -module A -bimodules is denoted by ${}^H_A\mathcal{M}_A$. We define morphisms in this category to be right A -module morphisms but not necessarily H -module morphisms or left A -module morphisms. The subcategory of ${}^H_A\mathcal{M}_A$ given by modules that are also H -module algebras, and morphisms that are algebra morphisms is denoted by ${}^H_A\mathcal{A}_A$.

In commutative differential geometry, as discussed in the introduction, we encounter all these structures. For example we have that the bimodule of one forms Ω on a manifold M is an element in the category ${}^{U\Xi}_{C^\infty(M)}\mathcal{M}_{C^\infty(M)}$, where $U\Xi$ is the universal enveloping algebra of the Lie algebra of vector fields and their action is given by the Lie derivative.

Following Drinfeld [3] and Giaquinto and Zhang [10] we now deform these modules and algebras via a twist

Definition 2.2. A **twist** $\mathcal{F} \in H \otimes H$ of a Hopf algebra H is an invertible element that satisfies

$$\mathcal{F}_{12}(\Delta \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta)\mathcal{F}, \quad (2\text{-cocycle property}) \quad (2.5a)$$

$$(\varepsilon \otimes \text{id})\mathcal{F} = 1 = (\text{id} \otimes \varepsilon)\mathcal{F}, \quad (\text{normalization property}) \quad (2.5b)$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$.

We shall frequently use the notation (sum over α understood)

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha. \quad (2.6)$$

In order to get familiar with this notation we rewrite the inverse of the 2-cocycle condition (2.5a), $((\Delta \otimes \text{id})\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} = ((\text{id} \otimes \Delta)\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}$, in this notation,

$$\bar{f}_1^\alpha \bar{f}^\beta \otimes \bar{f}_2^\alpha \bar{f}_\beta \otimes \bar{f}_\alpha = \bar{f}^\alpha \otimes \bar{f}_{\alpha_1} \bar{f}^\beta \otimes \bar{f}_{\alpha_2} \bar{f}_\beta. \quad (2.7)$$

In deformation quantization the twist is $\mathcal{F} = 1 \otimes 1 + hF_1 + h^2F_2 + \dots$ where $F_i \in H \otimes H$, and leads to a formal deformation quantization in the parameter h , see Theorem 2.4.

Theorem 2.3. The twist \mathcal{F} of the Hopf algebra H leads to a new Hopf algebra $H^\mathcal{F}$, given by

$$(H, \mu, \Delta^\mathcal{F}, S^\mathcal{F}, \varepsilon). \quad (2.8)$$

As algebras $H^\mathcal{F} = H$ and they also have the same counit $\varepsilon^\mathcal{F} = \varepsilon$. The new coproduct $\Delta^\mathcal{F}$ is given by, for all $\xi \in H$, $\Delta^\mathcal{F}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}$. The new antipode is $S^\mathcal{F}(\xi) = \chi S(\xi)\chi^{-1}$, where $\chi := f^\alpha S(f_\alpha)$, $\chi^{-1} = S(\bar{f}^\alpha)\bar{f}_\alpha$.

Theorem 2.4. Given a Hopf algebra H , a twist $\mathcal{F} \in H \otimes H$ and a left H -module algebra A (not necessarily associative or with unit), then there exists a deformed left $H^\mathcal{F}$ -module algebra A_\star . The algebra A_\star has the same \mathbb{K} -module structure as A and the action of $H^\mathcal{F}$ on A_\star is that of H on A . The product in A_\star is defined by, for all $a, b \in A$,

$$a \star b := \mu \circ \mathcal{F}^{-1} \triangleright (a \otimes b) = (\bar{f}^\alpha \triangleright a) (\bar{f}_\alpha \triangleright b). \quad (2.9)$$

If A has a unit element then A_\star has the same unit element. If A is associative then A_\star is an associative algebra as well.

Proof. We have to prove that the product in A_\star is compatible with the Hopf algebra structure on $H^\mathcal{F}$, for all $a, b \in A$ and $\xi \in H$,

$$\begin{aligned} \xi \triangleright (a \star b) &= \xi \triangleright (\mu \circ \mathcal{F}^{-1} \triangleright (a \otimes b)) = \mu \circ \Delta(\xi) \triangleright \circ \mathcal{F}^{-1} \triangleright (a \otimes b) = \mu \circ (\Delta(\xi) \mathcal{F}^{-1}) \triangleright (a \otimes b) \\ &= \mu \circ \mathcal{F}^{-1} \triangleright \circ \Delta^\mathcal{F}(\xi) \triangleright (a \otimes b) = (\xi_{1\mathcal{F}} \triangleright a) \star (\xi_{2\mathcal{F}} \triangleright b), \end{aligned} \quad (2.10)$$

where we used the notation $\Delta^\mathcal{F}(\xi) = \xi_{1\mathcal{F}} \otimes \xi_{2\mathcal{F}}$.

If A has a unit element 1 , then $1 \star a = a \star 1 = a$ follows from the normalization property (2.5b) of the twist \mathcal{F} . If A is an associative algebra we also have to prove associativity of the new product, for all $a, b, c \in A$,

$$\begin{aligned} (a \star b) \star c &= \bar{f}^\alpha \triangleright ((\bar{f}^\beta \triangleright a)(\bar{f}_\beta \triangleright b)) (\bar{f}_\alpha \triangleright c) \\ &= (\bar{f}^\alpha \triangleright a) \bar{f}_\alpha \triangleright ((\bar{f}^\beta \triangleright b)(\bar{f}_\beta \triangleright c)) = a \star (b \star c) , \end{aligned} \quad (2.11)$$

where we used the 2-cocycle property (2.5a) of the twist in the notation of (2.7). \square

Theorem 2.5. *In the hypotheses of Theorem 2.4, given a left H -module A -bimodule $V \in {}^H_A \mathcal{M}_A$, then there exists a left $H^\mathcal{F}$ -module A_\star -bimodule $V_\star \in {}^{H^\mathcal{F}}_{A_\star} \mathcal{M}_{A_\star}$. The module V_\star has the same \mathbb{K} -module structure as V and the left action of $H^\mathcal{F}$ on V_\star is that of H on V . The A_\star actions on V_\star are respectively defined by, for all $a \in A$ and $v \in V$,*

$$a \star v = \cdot \circ \mathcal{F}^{-1} \triangleright (a \otimes v) = (\bar{f}^\alpha \triangleright a) \cdot (\bar{f}_\alpha \triangleright v) , \quad (2.12a)$$

$$v \star a = \cdot \circ \mathcal{F}^{-1} \triangleright (v \otimes a) = (\bar{f}^\alpha \triangleright v) \cdot (\bar{f}_\alpha \triangleright a) . \quad (2.12b)$$

If V is further a left H -module A -bimodule algebra $V = E \in {}^H_A \mathcal{A}_A$, then $E_\star \in {}^{H^\mathcal{F}}_{A_\star} \mathcal{A}_{A_\star}$, where the \star -product in the algebra E_\star is given in Theorem 2.4.

Proof. Left A_\star -module property:

$$(a \star b) \star v = \bar{f}^\alpha \triangleright ((\bar{f}^\beta \triangleright a)(\bar{f}_\beta \triangleright b)) \cdot (\bar{f}_\alpha \triangleright v) = (\bar{f}^\alpha \triangleright a) \cdot \bar{f}_\alpha \triangleright ((\bar{f}^\beta \triangleright b) \cdot (\bar{f}_\beta \triangleright v)) = a \star (b \star v) . \quad (2.13)$$

The right A_\star -module and A_\star -bimodule properties are similarly proven. Compatibility between the left $H^\mathcal{F}$ and the left A_\star -action, $\xi \triangleright (a \star v) = (\xi_{1\mathcal{F}} \triangleright a) \star (\xi_{2\mathcal{F}} \triangleright v)$, is proven as in (2.10). Also the left $H^\mathcal{F}$ and the right A_\star -action compatibility is similarly proven.

In case we have $V = E \in {}^H_A \mathcal{A}_A$, then $E_\star \in {}^{H^\mathcal{F}}_{A_\star} \mathcal{A}_{A_\star}$ because of Theorem 2.4. \square

We observe that these structures can be untwisted to the original ones. This is done using the twist \mathcal{F}^{-1} of the Hopf algebra $H^\mathcal{F}$. Moreover, if we consider only H -equivariant morphisms then these are not deformed by the twisting procedure. Then Theorem 2.5 states that the category ${}^H_A \mathcal{M}_A^{eqv}$ and the deformed category ${}^{H^\mathcal{F}}_{A_\star} \mathcal{M}_{A_\star}^{eqv}$ of A -bimodules with H -equivariant morphisms are equivalent [10]; (this result follows from the equivalence of the tensor categories of H -modules and twisted H -modules [3]).

2.2 Twisting morphisms: the quantization map $D_\mathcal{F}$

The action \triangleright of an Hopf algebra H on an H -module V lifts to $End_{\mathbb{K}}(V)$ (the algebra of endomorphisms or \mathbb{K} -linear maps $V \rightarrow V$), via the adjoint action, for all $\xi \in H$ and $P \in End_{\mathbb{K}}(V)$,

$$\xi \blacktriangleright P := \xi_1 \triangleright \circ P \circ S(\xi_2) \triangleright . \quad (2.14)$$

This gives the algebra of endomorphisms $End_{\mathbb{K}}(V)$ an H -module algebra structure. We can consider the deformed algebra $End_{\mathbb{K}}(V)_\star$, with the new composition product between endomorphisms $P, Q \in End(V)$,

$$P \circ_\star Q := (\bar{f}^\alpha \blacktriangleright P) \circ (\bar{f}_\alpha \blacktriangleright Q) . \quad (2.15)$$

The algebras $End_{\mathbb{K}}(V)_\star$ and $End_{\mathbb{K}}(V)$ are respectively $H^\mathcal{F}$ and H -module algebras. However as algebras they are isomorphic. This was proven in [2] in the case A is H itself. The same techniques show that it holds more in general [11, 4]. The isomorphism is given by

$$D_\mathcal{F} : End(V)_\star \rightarrow End(V) , \quad P \mapsto D_\mathcal{F}(P) := (\bar{f}^\alpha \blacktriangleright P) \circ \bar{f}_\alpha \triangleright = \bar{f}_1^\alpha \triangleright \circ P \circ S(\bar{f}_2^\alpha) \bar{f}_\alpha \triangleright . \quad (2.16)$$

The twist and Hopf algebras properties imply that, for all $P, Q \in End(V)$

$$D_\mathcal{F}(P \circ_\star Q) = D_\mathcal{F}(P) \circ D_\mathcal{F}(Q) , \quad (2.17)$$

or equivalently $D_\mathcal{F} \circ \mu \circ \mathcal{F}^{-1} \triangleright = \mu \circ (D_\mathcal{F} \otimes D_\mathcal{F})$ (where $\mu(P \otimes Q) = P \circ Q$). An equivalent expression for $D_\mathcal{F}$ can be shown to be $D_\mathcal{F}(P) = f^\beta \triangleright \circ P \circ S(f_\beta) \chi^{-1} \triangleright$, where $\chi^{-1} = S(\bar{f}^\alpha) \bar{f}_\alpha$. The inverse of $D_\mathcal{F}$ is then given by $D_\mathcal{F}^{-1}(P) = \bar{f}^\alpha \triangleright \circ P \circ \chi S(\bar{f}_\alpha) \triangleright$, where $\chi = f^\beta S(f_\beta)$.

Remark 2.6. We mention a special example in which V is the algebra $A = C^\infty(M)$ on a manifold M , and H is the universal enveloping algebra $U\Xi$ of vector fields. Functions $f \in A$ can be seen as \mathbb{K} -linear maps $f : A \rightarrow A$, $h \mapsto fh$ for all $h \in A$. Then formula (2.17) reads, for $f, g \in A$,

$$D_{\mathcal{F}}(f \star g) = D_{\mathcal{F}}(f) \circ D_{\mathcal{F}}(g) \quad \text{i.e.,} \quad f \star g = D_{\mathcal{F}}^{-1}(D_{\mathcal{F}}(f) \circ D_{\mathcal{F}}(g)) . \quad (2.18)$$

In words: the star product of functions can be obtained by mapping these to the differential operators $D_{\mathcal{F}}(f)$ and $D_{\mathcal{F}}(g)$, composing, and then transforming back to function space.

There is another route to the quantization of the H -module $\text{End}_{\mathbb{K}}(V)$. We first consider V as an $H^{\mathcal{F}}$ -module (this is trivially so because H and $H^{\mathcal{F}}$ are the same algebra), however, to stress that now it belongs to the $H^{\mathcal{F}}\mathcal{M}$ category we denote it by V_{\star} . Then $\text{End}_{\mathbb{K}}(V_{\star}) \in H^{\mathcal{F}}\mathcal{M}$, where the $H^{\mathcal{F}}$ adjoint action is given by, for all $\xi \in H^{\mathcal{F}}$ and $P \in \text{End}_{\mathbb{K}}(V_{\star})$,

$$\xi \blacktriangleright_{\mathcal{F}} P := \xi_{1\mathcal{F}} \triangleright \circ P \circ S^{\mathcal{F}}(\xi_{2\mathcal{F}}) \triangleright . \quad (2.19)$$

We will frequently write $(\text{End}_{\mathbb{K}}(V_{\star}), \blacktriangleright_{\mathcal{F}}) \in H^{\mathcal{F}}\mathcal{M}$ in order to specify the $H^{\mathcal{F}}$ -action we are considering.

Theorem 2.7. *The map*

$$\begin{aligned} D_{\mathcal{F}} : \text{End}_{\mathbb{K}}(V)_{\star} &\longrightarrow \text{End}_{\mathbb{K}}(V_{\star}) \\ P &\longmapsto D_{\mathcal{F}}(P) := (\bar{f}^{\alpha} \blacktriangleright P) \circ \bar{f}_{\alpha} \triangleright \end{aligned} \quad (2.20)$$

is an isomorphism between the left $H^{\mathcal{F}}$ -module algebras $(\text{End}_{\mathbb{K}}(V)_{\star}, \blacktriangleright) \in H^{\mathcal{F}}\mathcal{A}$ and $(\text{End}_{\mathbb{K}}(V_{\star}), \blacktriangleright_{\mathcal{F}}) \in H^{\mathcal{F}}\mathcal{A}$. We call $D_{\mathcal{F}}(P)$ the quantization of the endomorphism P .

Proof. We already know that $D_{\mathcal{F}}$ is an isomorphism of algebras. We have to check $H^{\mathcal{F}}$ -equivariance, i.e., that $D_{\mathcal{F}}$ intertwines between the two $H^{\mathcal{F}}$ -actions, for all $\xi \in H$ and $P \in \mathbb{A}$,

$$D_{\mathcal{F}}(\xi \triangleright P) = \xi \triangleright_{\mathcal{F}} D_{\mathcal{F}}(P) . \quad (2.21)$$

Using the expression $D_{\mathcal{F}}(\xi \triangleright P) = f^{\beta}(\xi \triangleright P)S(f_{\beta})\chi^{-1}$ we have

$$\begin{aligned} D_{\mathcal{F}}(\xi \triangleright P) &= f^{\beta}(\xi \triangleright P)S(f_{\beta})\chi^{-1} = f^{\beta}\xi_1 PS(\xi_2)S(f_{\beta})\chi^{-1} = f^{\beta}\xi_1 \bar{f}^{\gamma} f^{\delta} PS(f_{\beta}\xi_2 \bar{f}_{\gamma} f_{\delta})\chi^{-1} \\ &= \xi_{1\mathcal{F}} f^{\delta} PS(f_{\delta})\chi^{-1} \chi S(\xi_{2\mathcal{F}})\chi^{-1} = \xi_{1\mathcal{F}} D_{\mathcal{F}}(P) S^{\mathcal{F}}(\xi_{2\mathcal{F}}) \\ &= \xi \triangleright_{\mathcal{F}} D_{\mathcal{F}}(P) , \end{aligned} \quad (2.22)$$

where in the third equality we inserted $1 \otimes 1 = \mathcal{F}^{-1}\mathcal{F}$. \square

Let $\text{Hom}_{\mathbb{K}}(V, W)$ denote the space of \mathbb{K} -linear maps (morphisms) from V to W . Similarly to $\text{End}_{\mathbb{K}}(V) \in H\mathcal{A}$ we have $\text{Hom}_{\mathbb{K}}(V, W) \in H\mathcal{M}$, and we can consider the quantizations $(\text{Hom}_{\mathbb{K}}(V, W)_{\star}, \blacktriangleright) \in H^{\mathcal{F}}\mathcal{M}$, and $(\text{Hom}_{\mathbb{K}}(V_{\star}, W_{\star}), \blacktriangleright_{\mathcal{F}}) \in H^{\mathcal{F}}\mathcal{M}$; here we have explicitly written the adjoint $H^{\mathcal{F}}$ -actions carried by the two modules. Then, as in Theorem 2.7, the quantization map is an isomorphisms between these two $H^{\mathcal{F}}$ -modules. Also the composition $P \circ Q$ of two \mathbb{K} -linear maps $Q : Z \rightarrow V$ and $P : V \rightarrow W$ can be deformed in the \star -composition $P \circ_{\star} Q = (\bar{f}^{\alpha} \blacktriangleright P) \circ (\bar{f}_{\alpha} \blacktriangleright Q)$.

These data have a categorical description. We have three categories: $(H\mathcal{M}, \circ, \blacktriangleright)$, and the twisted ones $(H^{\mathcal{F}}\mathcal{M}, \circ_{\star}, \blacktriangleright)$ and $(H^{\mathcal{F}}\mathcal{M}, \circ, \blacktriangleright_{\mathcal{F}})$.

In $(H\mathcal{M}, \circ, \blacktriangleright)$, objects are H -modules and morphisms are \mathbb{K} -linear maps with their usual composition. These maps are not H -equivariant but carry a specific H -action \blacktriangleright , the one in (2.14), that is canonically lifted from the H -action on the (source and target) modules. This action is compatible with composition of morphism, for all $\xi \in H$, $P : V \rightarrow W$, $Q : Z \rightarrow V$, $\xi \blacktriangleright (P \circ Q) = (\xi_1 \blacktriangleright P) \circ (\xi_2 \blacktriangleright Q)$.

In $({}^H\mathcal{M}, \circ_\star, \blacktriangleright)$, objects are $H^\mathcal{F}$ -modules and morphisms are \mathbb{K} -linear maps with \star -composition. These maps carry the same \blacktriangleright action (2.14), that now is seen as an $H^\mathcal{F}$ -action (this is doable since H and $H^\mathcal{F}$ are the same as algebras). This $H^\mathcal{F}$ -action is compatible with \star -composition of morphisms, $\xi \blacktriangleright (P \circ_\star Q) = (\xi_{1\mathcal{F}} \blacktriangleright P) \circ_\star (\xi_{2\mathcal{F}} \blacktriangleright Q)$.

In $({}^H\mathcal{M}, \circ, \blacktriangleright_\mathcal{F})$, objects are $H^\mathcal{F}$ -modules and morphisms are \mathbb{K} -linear maps with their usual composition. The $H^\mathcal{F}$ -action $\blacktriangleright_\mathcal{F}$ on these maps is the (2.19) one, that is canonically lifted from the $H^\mathcal{F}$ -action on the (source and target) modules. This action is compatible with composition of morphisms, $\xi \blacktriangleright_\mathcal{F} (P \circ Q) = (\xi_{1\mathcal{F}} \blacktriangleright_\mathcal{F} P) \circ (\xi_{2\mathcal{F}} \blacktriangleright_\mathcal{F} Q)$.

It follows from Theorem 2.7 that $({}^H\mathcal{M}, \circ_\star, \blacktriangleright)$ and $({}^H\mathcal{M}, \circ, \blacktriangleright_\mathcal{F})$ are equivalent categories via the functor that is the identity on objects and $D_\mathcal{F}$ on morphisms. Indeed $D_\mathcal{F}$ satisfies (2.17) and (2.21) for P, Q composable morphisms.

2.3 Twisting A -module morphisms

The findings of the previous section holds also in the subcategory ${}^H_A\mathcal{M}_A$ of H -modules A -bimodules, where A is an H -module algebra. In this case morphisms are right A -linear maps, we use the notations $\text{Hom}_A(V, W)$ and $\text{End}_A(V)$ for right A -linear morphisms and endomorphisms (where $V, W \in {}^H_A\mathcal{M}_A$).

The left A -module structure of V represents A as right A -module endomorphisms of V , for all $a \in A$, $a \mapsto l_a \in \text{End}_A(V)$, where $l_a(v) = a \cdot v$ for all $v \in V$. Then $\text{End}_A(V)$ is an A -bimodule by defining, for all $P \in \text{End}_A(V)$,

$$a \cdot P = l_a \circ P \quad , \quad P \cdot a = P \circ l_a \quad . \quad (2.23)$$

This A -bimodule structure is compatible with the H -module one given by the adjoint action \blacktriangleright , so that $(\text{End}_A(V), \blacktriangleright) \in {}^H_A\mathcal{A}_A$. Similarly $(\text{Hom}_A(V, W), \blacktriangleright) \in {}^H_A\mathcal{M}_A$.

Theorem 2.8. *The map*

$$\begin{aligned} D_\mathcal{F} : \text{End}_A(V)_\star &\longrightarrow \text{End}_A(V_\star) \\ P &\longmapsto D_\mathcal{F}(P) := (\bar{f}^\alpha \blacktriangleright P) \circ \bar{f}_\alpha \triangleright \end{aligned} \quad (2.24)$$

is an isomorphism between the left $H^\mathcal{F}$ -module A_\star -bimodule algebras $(\text{End}_A(V)_\star, \blacktriangleright) \in {}^{H^\mathcal{F}}_{A_\star}\mathcal{A}_{A_\star}$ and $(\text{End}_{A_\star}(V_\star), \blacktriangleright_\mathcal{F}) \in {}^{H^\mathcal{F}}_{A_\star}\mathcal{A}_{A_\star}$.

3 Twisting and tensoring

3.1 Quasitriangular Hopf algebras and tensor product of \mathbb{K} -linear maps

The tensor product $V \otimes W$ of two left H -modules $V, W \in {}^H\mathcal{M}$ over the Hopf algebra H is again a left H -module, $V \otimes W \in {}^H\mathcal{M}$. The left H -action is defined using the coproduct, for all $\xi \in H$, $v \in V$ and $w \in W$,

$$\xi \triangleright (v \otimes w) := (\xi_1 \triangleright v) \otimes (\xi_2 \triangleright w) \quad , \quad (3.1)$$

and extended by linearity to all $V \otimes W$.

Given two linear maps $V \xrightarrow{P} \widetilde{V}$ and $W \xrightarrow{Q} \widetilde{W}$, the tensor product map $V \otimes W \xrightarrow{P \otimes Q} \widetilde{V} \otimes \widetilde{W}$ is defined by, for all $v \in V, w \in W$,

$$(P \otimes Q)(v \otimes w) = P(v) \otimes Q(w) \quad (3.2)$$

and extended by linearity to all $V \otimes W$.

This tensor product is not compatible with the H -action, indeed a short calculation shows that, for all $\xi \in H$, $\xi \blacktriangleright (P \otimes Q) \neq (\xi_1 \blacktriangleright P) \otimes (\xi_2 \blacktriangleright Q)$; equality holding in case Q is H -equivariant i.e., $\xi \blacktriangleright Q = \varepsilon(\xi)Q$. In order to introduce a tensor product compatible with the H -action we need a **quasitriangular Hopf**

algebra (H, \mathcal{R}) . This is a Hopf algebra H with an invertible element $\mathcal{R} \in H \otimes H$ (called universal R -matrix). We use the notation $\mathcal{R} = R^\alpha \otimes R_\alpha$. Let $\mathcal{R}_{21} = R_\alpha \otimes R^\alpha$ then if $\mathcal{R}_{21} = \mathcal{R}^{-1}$, the quasitriangular Hopf algebra (H, \mathcal{R}) is called **triangular**. Among the properties of the \mathcal{R} -matrix we recall that it satisfies the Yang-Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

The \mathcal{R} -matrix induces an isomorphism, called **braiding**, between the tensor product modules $V \otimes W \in {}^H\mathcal{M}$ and $W \otimes V \in {}^H\mathcal{M}$,

$$\tau_{\mathcal{R} \, W, V} : W \otimes V \rightarrow V \otimes W, \quad w \otimes v \mapsto \tau_{\mathcal{R} \, W, V}(w \otimes v) = (\overline{R}^\alpha \triangleright v) \otimes (\overline{R}_\alpha \triangleright w), \quad (3.3)$$

where we used the notation $\mathcal{R}^{-1} = \overline{R}^\alpha \otimes \overline{R}_\alpha$.

The map $\tau_{\mathcal{R} \, W, V}$ is a left H -module isomorphisms, i.e., for all $\xi \in H, v \in V, w \in W$, $\xi \triangleright (\tau_{\mathcal{R}}(w \otimes v)) = \tau_{\mathcal{R}}(\xi \triangleright (w \otimes v))$, or equivalently it is H -equivariant $\xi \blacktriangleright \tau_{\mathcal{R}} = \varepsilon(\xi) \tau_{\mathcal{R}}$. Let $V, W, Z \in {}^H\mathcal{M}$ then it follows from the \mathcal{R} -matrix properties that on the triple tensor product $V \otimes W \otimes Z$ we have the braid relations. For later purposes we write them in terms of the inverse braid τ^{-1} ,

$$\tau_{\mathcal{R} \, 1(23)}^{-1} = \tau_{\mathcal{R} \, 23}^{-1} \circ \tau_{\mathcal{R} \, 12}^{-1} \quad (3.4a)$$

$$\tau_{\mathcal{R} \, (12)3}^{-1} = \tau_{\mathcal{R} \, 12}^{-1} \circ \tau_{\mathcal{R} \, 23}^{-1} \quad (3.4b)$$

where $\tau_{\mathcal{R} \, 12}^{-1}$ acts on the first and second entry of the triple tensor product, $\tau_{\mathcal{R} \, 23}^{-1}$ on the second and third, and $\tau_{\mathcal{R} \, 1(23)}^{-1}$ (or $\tau_{\mathcal{R} \, V, W \otimes Z}^{-1}$) exchanges the first entry with the second and third ones.

The first expression for example states that flipping an element v to the right of the element $w \otimes z$ is the same as first flipping v to the right of w and then the result to the right of z .

Example 3.1. The universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra \mathfrak{g} is a cocommutative Hopf algebra (i.e., for all $\xi \in H$, $\Delta(\xi) = \xi_1 \otimes \xi_2 = \xi_2 \otimes \xi_1$). Every cocommutative Hopf algebra H has a triangular structure given by the R -matrix $\mathcal{R} = 1 \otimes 1$. Let \mathcal{F} be a twist of this cocommutative Hopf algebra H , then the Hopf algebra $H^\mathcal{F}$ is triangular with \mathcal{R} -matrix $\mathcal{R}^\mathcal{F} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

We can now define the \mathcal{R} -tensor product of \mathbb{K} -linear maps, see also [12] Chapter 9.3.

Definition 3.2. Let (H, \mathcal{R}) be a quasitriangular Hopf algebra and $V, W, \widetilde{V}, \widetilde{W} \in {}^H\mathcal{M}$ be left H -modules. The **\mathcal{R} -tensor product** of \mathbb{K} -linear maps is defined by, for all $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$ and $Q \in \text{Hom}_{\mathbb{K}}(W, \widetilde{W})$,

$$P \otimes_{\mathcal{R}} Q := (P \circ \overline{R}^\alpha \triangleright) \otimes (\overline{R}_\alpha \blacktriangleright Q) \in \text{Hom}_{\mathbb{K}}(V \otimes W, \widetilde{V} \otimes \widetilde{W}), \quad (3.5)$$

where \otimes is defined in (3.2).

From the definition it immediately follows that

$$P \otimes_{\mathcal{R}} Q = (P \otimes \text{id}) \circ (\overline{R}^\alpha \triangleright \otimes \overline{R}_\alpha \blacktriangleright Q) = (P \otimes_{\mathcal{R}} \text{id}) \circ (\text{id} \otimes_{\mathcal{R}} Q). \quad (3.6)$$

The lift of $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$ is simply $P \otimes_{\mathcal{R}} \text{id} = P \otimes \text{id}$, while the lift of Q is

$$\text{id} \otimes_{\mathcal{R}} Q = \overline{R}^\alpha \triangleright \otimes \overline{R}_\alpha \blacktriangleright Q. \quad (3.7)$$

Use of the braiding map $\tau_{\mathcal{R}}$ (cf. (3.3)) allows us to rewrite the lift $\text{id} \otimes_{\mathcal{R}} Q$ acting on $V \otimes W$ in terms of the lift $Q \otimes \text{id}$ acting on $W \otimes V$, $\text{id} \otimes_{\mathcal{R}} Q = \tau_{\mathcal{R}} \circ (Q \otimes \text{id}) \circ \tau_{\mathcal{R}}^{-1}$.

We summarize the properties of the \mathcal{R} -tensor product $\otimes_{\mathcal{R}}$ in the following

Theorem 3.3. Let (H, \mathcal{R}) be a quasitriangular Hopf algebra and $V, W, Z, \widetilde{V}, \widetilde{W}, \widehat{Z}, \widehat{V}, \widehat{W} \in {}^H\mathcal{M}$ be left H -modules. The \mathcal{R} -tensor product is compatible with the left H -module structure, i.e., for all $\xi \in H$, $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$ and $Q \in \text{Hom}_{\mathbb{K}}(W, \widetilde{W})$,

$$\xi \blacktriangleright (P \otimes_{\mathcal{R}} Q) = (\xi_1 \blacktriangleright P) \otimes_{\mathcal{R}} (\xi_2 \blacktriangleright Q). \quad (3.8a)$$

Furthermore, the \mathcal{R} -tensor product is associative, i.e., for all $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$, $Q \in \text{Hom}_{\mathbb{K}}(W, \widetilde{W})$ and $T \in \text{Hom}_{\mathbb{K}}(Z, \widetilde{Z})$,

$$(P \otimes_{\mathcal{R}} Q) \otimes_{\mathcal{R}} T = P \otimes_{\mathcal{R}} (Q \otimes_{\mathcal{R}} T) , \quad (3.8b)$$

and satisfies the composition law, for all $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$, $Q \in \text{Hom}_{\mathbb{K}}(W, \widetilde{W})$, $\tilde{P} \in \text{Hom}_{\mathbb{K}}(\widetilde{V}, \widehat{V})$ and $\tilde{Q} \in \text{Hom}_{\mathbb{K}}(\widetilde{W}, \widehat{W})$,

$$(\tilde{P} \otimes_{\mathcal{R}} \tilde{Q}) \circ (P \otimes_{\mathcal{R}} Q) = (\tilde{P} \circ (\overline{R}^\alpha \blacktriangleright P)) \otimes_{\mathcal{R}} ((\overline{R}_\alpha \blacktriangleright \tilde{Q}) \circ Q) . \quad (3.8c)$$

We have studied the tensor product of modules \otimes , and that of morphisms $\otimes_{\mathcal{R}}$ in the category of H -modules; that henceforth we denote by $({}^H\mathcal{M}, \circ, \blacktriangleright, \otimes, \otimes_{\mathcal{R}})^2$.

3.2 Twisting tensor product modules and morphisms

In the same way we have constructed the category $({}^H\mathcal{M}, \circ, \blacktriangleright, \otimes, \otimes_{\mathcal{R}})$, where modules are denoted by V, W, \dots , morphisms by P, Q, \dots and their tensor products by $V \otimes W$, $P \otimes_{\mathcal{R}} Q$, we can construct the category $({}^H\mathcal{M}, \circ, \blacktriangleright_{\mathcal{F}}, \otimes_{\star}, \otimes_{\mathcal{R}^{\mathcal{F}}})$. We have just to replace the quasitriangular Hopf algebra (H, \mathcal{R}) with the quasitriangular one $(H^{\mathcal{F}}, \mathcal{R}^{\mathcal{F}})$, with universal \mathcal{R} -matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1}$.

In $({}^H\mathcal{M}, \circ, \blacktriangleright_{\mathcal{F}}, \otimes_{\star}, \otimes_{\mathcal{R}^{\mathcal{F}}})$ we denote $H^{\mathcal{F}}$ -modules by $V_{\star}, W_{\star}, \dots$ (recall however that as \mathbb{K} -modules they are the same as V, W, \dots), morphisms, that are \mathbb{K} -linear maps, by P, Q, \dots (for a \mathbb{K} -linear map $V_{\star} \xrightarrow{P} W_{\star}$ we have $V = V_{\star}, W = W_{\star}$, the H - or $H^{\mathcal{F}}$ -module structure being irrelevant), and their tensor products by $V_{\star} \otimes_{\star} W_{\star}$, $P \otimes_{\mathcal{R}^{\mathcal{F}}} Q$. The notation $V_{\star} \otimes_{\star} W_{\star}$ stresses that the $H^{\mathcal{F}}$ -action on $V_{\star} \otimes_{\star} W_{\star}$ is obtained using the $H^{\mathcal{F}}$ -coproduct (not the H -one), explicitly $\xi \triangleright (v \otimes_{\star} w) = (\xi_{1\mathcal{F}} \triangleright v) \otimes_{\star} (\xi_{2\mathcal{F}} \triangleright w)$, for all $\xi \in H^{\mathcal{F}}$, $v \in V_{\star}, w \in W_{\star}$.

Given the H -modules V, W we can then consider the $H^{\mathcal{F}}$ -modules $V_{\star} \otimes_{\star} W_{\star}$ and $(V \otimes W)_{\star}$. The $H^{\mathcal{F}}$ action on these modules is different because on $(V \otimes W)_{\star}$ it is obtained using the H -coproduct (not the $H^{\mathcal{F}}$ -one), recall (3.1). It is easy to show that $V_{\star} \otimes_{\star} W_{\star}$ and $(V \otimes W)_{\star}$ are isomorphic via the \mathbb{K} -linear and H -equivariant map

$$\varphi_{V,W} := \mathcal{F}^{-1} \triangleright : V_{\star} \otimes_{\star} W_{\star} \rightarrow (V \otimes W)_{\star} , \quad v \otimes_{\star} w \mapsto \varphi_{V,W}(v \otimes_{\star} w) = (\overline{f}^\alpha \triangleright v) \otimes (\overline{f}_\alpha \triangleright w) . \quad (3.9)$$

Indeed, $\varphi_{V,W}(\xi \triangleright_{\mathcal{F}} (v \otimes_{\star} w)) = \xi \triangleright (\varphi_{V,W}(v \otimes_{\star} w))$, and the inverse of $\varphi_{V,W}$ is $\varphi_{V,W}^{-1} = \mathcal{F} \triangleright$.

Twist quantization of tensor products of morphisms is then described by the following

Theorem 3.4. *Let (H, \mathcal{R}) be a quasitriangular Hopf algebra with twist $\mathcal{F} \in H \otimes H$ and $V, W, \widetilde{V}, \widetilde{W} \in {}^H\mathcal{M}$. Then for all $P \in \text{Hom}_{\mathbb{K}}(V, \widetilde{V})$ and $Q \in \text{Hom}_{\mathbb{K}}(W, \widetilde{W})$ the following diagram of \mathbb{K} -linear maps commutes:*

$$\begin{array}{ccc} V_{\star} \otimes_{\star} W_{\star} & \xrightarrow{D_{\mathcal{F}}(P) \otimes_{\mathcal{R}^{\mathcal{F}}} D_{\mathcal{F}}(Q)} & \widetilde{V} \otimes_{\star} \widetilde{W}_{\star} \\ \varphi_{V,W} \downarrow & & \downarrow \varphi_{\widetilde{V}, \widetilde{W}} \\ (V \otimes W)_{\star} & \xrightarrow{D_{\mathcal{F}}((\overline{f}^\alpha \blacktriangleright P) \otimes_{\mathcal{R}} (\overline{f}_\alpha \blacktriangleright Q))} & (\widetilde{V} \otimes \widetilde{W})_{\star} \end{array} \quad (3.10)$$

We notice that if P, Q are H -equivariant maps, then $\otimes_{\mathcal{R}^{\mathcal{F}}} = \otimes_{\mathcal{R}} = \otimes$ and the horizontal maps in (3.10) become simply $P \otimes Q$. Then (3.10) with H -equivariant maps P, Q , and commutativity of the diagram

$$\begin{array}{ccc} V_{\star} \otimes_{\star} W_{\star} \otimes_{\star} \Omega_{\star} & \xrightarrow{\varphi_{V,W} \otimes_{\mathcal{R}} \text{id}_{\Omega_{\star}}} & (V \otimes W)_{\star} \otimes_{\star} \Omega_{\star} \\ \text{id}_{V_{\star}} \otimes_{\mathcal{R}} \varphi_{W, \Omega} \downarrow & & \downarrow \varphi_{(V \otimes W), \Omega} \\ V_{\star} \otimes_{\star} (W \otimes \Omega)_{\star} & \xrightarrow{\varphi_{V, (W \otimes \Omega)}} & (V \otimes W \otimes \Omega)_{\star} \end{array} \quad (3.11)$$

²This category is not quite a tensor category because of (3.8c) (that shows that $(\otimes, \otimes_{\mathcal{R}})$ is not a bifunctor).

(that easily follows from the twist cocycle condition (2.7)) show that the categories of representations of the Hopf algebras H and $H^{\mathcal{F}}$ are equivalent.

In the more general case of arbitrary \mathbb{K} -linear maps P, Q , commutativity of the diagram (3.10) relates the categories $({}^{H^{\mathcal{F}}}\mathcal{M}, \circ, \blacktriangleright_{\mathcal{F}}, \otimes_{\star}, \otimes_{\mathcal{R}^{\mathcal{F}}})$ and $({}^{H^{\mathcal{F}}}\mathcal{M}, \circ_{\star}, \blacktriangleright, \otimes, \otimes_{\mathcal{R}_{\star}})$. In this latter category $({}^{H^{\mathcal{F}}}\mathcal{M}, \circ_{\star}, \blacktriangleright, \otimes, \otimes_{\mathcal{R}_{\star}})$ we denote modules by $V_{\star}, W_{\star}, \dots$ and morphisms by P, Q, \dots . The \otimes -tensor product of modules in this category is by definition the new module $(V \otimes W)_{\star}$. The tensor product of \mathbb{K} -linear maps is given by $P \otimes_{\mathcal{R}_{\star}} Q = (\bar{f}^{\alpha} \blacktriangleright P) \otimes_{\mathcal{R}} (\bar{f}_{\alpha} \blacktriangleright Q)$; it can be shown to be just (3.6) with the composition of morphisms \circ replaced by the \star -composition \circ_{\star} . [Hint: use (3.8a)].

3.3 Twisting tensor products of A -module morphisms

As in Subsection 2.3, we now consider the subcategory ${}^H_A\mathcal{M}_A$ of H -modules A -bimodules, where morphisms are right A -linear maps. In this subcategory we have the tensor product \otimes_A . We recall that the tensor product over A of the modules V and W , denoted $V \otimes_A W$, is the quotient of the \mathbb{K} -module $V \otimes W$ via the \mathbb{K} -submodule generated by the elements $v \cdot a \otimes w - v \otimes a \cdot w$, for all $a \in A, v \in V, w \in W$. The image of $v \otimes w$ under the canonical projection $\pi : V \otimes W \rightarrow V \otimes_A W$ is denoted by $v \otimes_A w$.

We further restrict to the case where A is a **quasi-commutative algebra**, i.e., by definition for all $a, \tilde{a} \in A$,

$$a \tilde{a} = (\bar{R}^{\alpha} \triangleright \tilde{a})(\bar{R}_{\alpha} \triangleright a) ; \quad (3.12)$$

and where the A -bimodules are **quasi-commutative**, i.e., by definition, for all $a \in A, v \in V$,

$$v \cdot a = (\bar{R}^{\alpha} \triangleright a) \cdot (\bar{R}_{\alpha} \triangleright v) . \quad (3.13)$$

Notice that in A we have $a \tilde{a} = (\bar{R}^{\alpha} \triangleright \tilde{a})(\bar{R}_{\alpha} \triangleright a) = (\bar{R}^{\beta} \bar{R}_{\alpha} \triangleright a)(\bar{R}_{\beta} \bar{R}^{\alpha} \triangleright \tilde{a})$. We recall that a triangular Hopf algebra (H, \mathcal{R}) , is a quasitriangular one with $\mathcal{R}^{-1} = \mathcal{R}_{21}$ i.e., $\bar{R}^{\beta} \bar{R}_{\alpha} \otimes \bar{R}_{\beta} \bar{R}^{\alpha} = 1 \otimes 1 \in H \otimes H$. We see that quasi-commutative modules and algebras are very natural in the context of triangular Hopf algebras. They naturally arise via twist quantization of commutative algebras that carry a representation of a cocommutative Hopf algebra with trivial \mathcal{R} -matrix $\mathcal{R} = 1 \otimes 1$, see Example 3.1 (see also [2]).

It can be shown that in quasi-commutative bimodules over a quasi-commutative algebra A , right A -linear maps are also **quasi-left A -linear**, for all $a \in A, w \in W, Q \in \text{Hom}_A(W, \widetilde{W})$,

$$Q(a \cdot w) = (\bar{R}^{\alpha} \triangleright a) \cdot (\bar{R}_{\alpha} \blacktriangleright Q)(w) . \quad (3.14)$$

Use of this property leads to prove that the $\otimes_{\mathcal{R}}$ -tensor product of \mathbb{K} -linear maps over $V \otimes W$ of Definition 3.2 induces an $\otimes_{\mathcal{R}}$ -tensor product of A -linear maps on the A -module $V \otimes_A W$, i.e., $V \otimes_A W \xrightarrow{P \otimes_{\mathcal{R}} Q} \widetilde{V} \otimes_A \widetilde{W}$ is a well defined right A -linear map if $V \xrightarrow{P} \widetilde{V}, W \xrightarrow{Q} \widetilde{W}$ are right A -linear and $V, \widetilde{V}, W, \widetilde{W} \in {}^H_A\mathcal{M}_A$ are quasi-commutative.

All the properties of the $\otimes_{\mathcal{R}}$ tensor product of \mathbb{K} -linear maps of Theorem 3.3 hold for this induced tensor product of A -linear maps.

Also the isomorphism $\varphi_{V,W} : V \otimes_{\star} W \rightarrow (V \otimes W)_{\star}$ induces the isomorphism on the quotient modules $\varphi_{V,W} : V \otimes_{A_{\star}} W \rightarrow (V \otimes_A W)_{\star}$, that satisfies a commutative diagram like (3.11).

Theorem 3.5. *Consider a quasi-commutative algebra $A \in {}^H\mathcal{A}$. Then Theorem 3.4 holds for right A -linear maps on quasi-commutative bimodules in ${}^H_A\mathcal{M}_A$. Just replace in diagram (3.10) the tensor products \otimes and \otimes_{\star} , with \otimes_A and $\otimes_{A_{\star}}$ respectively.*

4 Connections

4.1 Twisting connections

Let A be a unital and associative algebra over \mathbb{K} . A **differential calculus** $(\Omega^{\bullet}, \wedge, d)$ over A is a graded algebra $(\Omega^{\bullet} = \bigoplus_{n \geq 0} \Omega^n, \wedge)$ over \mathbb{K} , where $\Omega^0 = A$ has degree zero, together with \mathbb{K} -linear maps of degree

one $d : \Omega^n \rightarrow \Omega^{n+1}$, satisfying $d \circ d = 0$ and the graded Leibniz rule

$$d(\theta \wedge \theta') = (d\theta) \wedge \theta' + (-1)^{\deg(\theta)} \theta \wedge (d\theta') , \quad (4.1)$$

for all $\theta, \theta' \in \Omega^\bullet$, with θ of homogeneous degree. Because of the Leibniz rule, the \mathbb{K} -modules Ω^n , $n > 0$, are A -bimodules, i.e. $\Omega^n \in {}_A\mathcal{M}_A$. As in commutative differential geometry we call Ω^n the module of n -forms; we also assume that any 1-form $\theta \in \Omega := \Omega^1$ can be written as $\theta = \sum_i a_i db_i$, with $a_i, b_i \in A$, i.e. that exact 1-forms generate Ω as a left A -module.

Let $(\Omega^\bullet, \wedge, d)$ be a differential calculus over $A = \Omega^0$; a **right connection** on a bimodule $V \in {}_A\mathcal{M}_A$ is a \mathbb{K} -linear map $\nabla : V \rightarrow V \otimes_A \Omega$, satisfying the right Leibniz rule, for all $v \in V$ and $a \in A$,

$$\nabla(v \cdot a) = (\nabla v) \cdot a + v \otimes_A da . \quad (4.2)$$

We denote by $\text{Con}_A(V)$ the set of all connections on the bimodule $V \in {}_A\mathcal{M}_A$. Notice that this definition holds also if we just have a right A -module $V \in \mathcal{M}_A$; actually all the statements in this subsection hold in the category of right A -modules. $\text{Con}_A(V)$ is an affine space over $\text{Hom}_A(V, V \otimes_A \Omega)$ because for $\nabla \in \text{Con}_A(V)$ and $P \in \text{Hom}_A(V, V \otimes_A \Omega)$ then $\nabla + P \in \text{Con}_A(V)$, and any connection differs from a given one ∇ by a morphism P .

Let H be a Hopf algebra and let $A = \Omega^0$ and Ω^\bullet be left H -module algebras. The differential calculus $(\Omega^\bullet, \wedge, d)$ is a **left H -covariant differential calculus** over A , if the H -action \triangleright is degree preserving and the differential is equivariant, for all $\xi \in H$ and $\theta \in \Omega^\bullet$,

$$\xi \triangleright (d\theta) = d(\xi \triangleright \theta) . \quad (4.3)$$

Since the H -action is degree preserving we have for all $n \geq 0$, $\Omega^n \in {}^H_A\mathcal{M}_A$.

Given a twist $\mathcal{F} \in H \otimes H$, a left H -covariant differential calculus $(\Omega^\bullet, \wedge, d)$ over A can be quantized (see Theorem 2.4) to yield a left $H^\mathcal{F}$ -covariant differential calculus $(\Omega^\bullet, \wedge_\star, d)$ over A_\star (equivariance of the differential d implies that d is also a differential on $(\Omega^\bullet, \wedge_\star)$).

Let $V \in {}^H_A\mathcal{M}_A$; we now briefly outline how given a twist $\mathcal{F} \in H \otimes H$ of the Hopf algebra H , the quantization map $D_\mathcal{F}$ leads to an isomorphism $\text{Con}_A(V) \cong \text{Con}_{A_\star}(V_\star)$ between connections on the undeformed module $V \in {}^H_A\mathcal{M}_A$ and on the deformed module $V_\star \in {}^{H^\mathcal{F}}_{A_\star}\mathcal{M}_{A_\star}$.

We first observe that the left $H^\mathcal{F}$ -module A_\star -bimodule isomorphism $(V \otimes_A \Omega)_\star \xrightarrow{\varphi^{-1}} V_\star \otimes_{A_\star} \Omega_\star$ (where for ease of notation we dropped the module indices on φ) canonically leads to the isomorphism $\text{Hom}_\mathbb{K}(V_\star, (V \otimes_A \Omega)_\star) \xrightarrow{\varphi^{-1}} \text{Hom}_\mathbb{K}(V_\star, V_\star \otimes_{A_\star} \Omega_\star)$. Composition of the quantization map

$$D_\mathcal{F} : \text{Hom}_\mathbb{K}(V, V \otimes_A \Omega)_\star \longrightarrow \text{Hom}_\mathbb{K}(V_\star, (V \otimes_A \Omega)_\star) \quad (4.4)$$

with this isomorphism gives the left $H^\mathcal{F}$ -module A_\star -bimodule isomorphism

$$\tilde{D}_\mathcal{F} := \varphi^{-1} \circ D_\mathcal{F} : \text{Hom}_\mathbb{K}(V, V \otimes_A \Omega)_\star \longrightarrow \text{Hom}_\mathbb{K}(V_\star, V_\star \otimes_{A_\star} \Omega_\star) . \quad (4.5)$$

Theorem 4.1. *The isomorphism (4.5) restricts to the left $H^\mathcal{F}$ -module A_\star -bimodule isomorphism*

$$\tilde{D}_\mathcal{F} : \text{Hom}_A(V, V \otimes_A \Omega)_\star \longrightarrow \text{Hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega_\star) \quad , \quad P \longmapsto \varphi^{-1} \circ (\bar{f}^\alpha \blacktriangleright P) \circ \bar{f}_\alpha \triangleright \quad (4.6)$$

and to the affine space isomorphism

$$\tilde{D}_\mathcal{F} : \text{Con}_A(V) \longrightarrow \text{Con}_{A_\star}(V_\star) \quad , \quad \nabla \longmapsto \varphi^{-1} \circ (\bar{f}^\alpha \blacktriangleright \nabla) \circ \bar{f}_\alpha \triangleright \quad , \quad (4.7)$$

where $\text{Con}_A(V)$ and $\text{Con}_{A_\star}(V_\star)$ are respectively affine spaces over the isomorphic modules $\text{Hom}_A(V, V \otimes_A \Omega)_\star$ and $\text{Hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega_\star)$ of right A -linear, respectively A_\star -linear morphisms.

4.2 Connections on tensor product modules (sum of connections)

Connections on quasi-commutative bimodules can be summed to give connections on tensor product modules.

Let (H, \mathcal{R}) be a quasitriangular Hopf algebra. A left H -covariant differential calculus $(\Omega^\bullet, \wedge, d)$ over $A \in {}^H\mathcal{A}$ is called **graded quasi-commutative** if the algebra Ω^\bullet is graded quasi-commutative, i.e., for all $\theta, \theta' \in \Omega^\bullet$ of homogeneous degree,

$$\theta \wedge \theta' = (-1)^{\deg(\theta) \deg(\theta')} (\overline{R}^\alpha \triangleright \theta') \wedge (\overline{R}_\alpha \triangleright \theta) . \quad (4.8)$$

It can be shown that any left H -covariant differential calculus $(\Omega^\bullet, \wedge, d)$ over a quasi-commutative algebra A is graded quasi-commutative if the bimodule of one-forms Ω is quasi-commutative (and generates Ω^n for all $n > 1$).

Proposition 4.2. *Let (H, \mathcal{R}) be a quasi-triangular Hopf algebra and $(\Omega^\bullet, \wedge, d)$ be graded quasi-commutative. A right connection ∇ on a quasi-commutative A -bimodule $W \in {}^H\mathcal{M}_A$ is also a **quasi-left connection**, in the sense that we have the braided Leibniz rule, for all $a \in A$ and $w \in W$,*

$$\nabla(a \cdot w) = (\overline{R}^\alpha \triangleright a) \cdot (\overline{R}_\alpha \blacktriangleright \nabla)(w) + (R_\alpha \triangleright w) \otimes_A (R^\alpha \triangleright da) . \quad (4.9)$$

This property parallels the quasi-left A -linearity property (3.14) of right A -linear morphisms. Notice that if ∇ is H -equivariant we recover the notion of A -bimodule connection [6], [7] Section 3.6.

Lemma 4.3. *Let $W \in {}^H\mathcal{M}_A$ be quasi-commutative and $\Omega \in {}^H\mathcal{M}_A$ be the bimodule of one-forms of a left H -covariant graded quasi-commutative differential calculus $(\Omega^\bullet, \wedge, d)$. Then the inverse braiding map $\tau_{\mathcal{R}}^{-1} : \Omega \otimes W \rightarrow W \otimes \Omega$ (recall definition (3.3)) canonically induces a left H -module A -bimodule isomorphism on the quotient*

$$\tau_{\mathcal{R}}^{-1} : \Omega \otimes_A W \longrightarrow W \otimes_A \Omega \quad , \quad \theta \otimes_A w \mapsto \tau_{\mathcal{R}}^{-1}(\theta \otimes w) = (R_\alpha \triangleright w) \otimes_A (R^\alpha \triangleright \theta) . \quad (4.10)$$

This satisfies the braid relation $\tau_{\mathcal{R}1(23)}^{-1} = \tau_{\mathcal{R}23}^{-1} \circ \tau_{\mathcal{R}12}^{-1}$ that is induced from the braid relation (3.4a).

Given two connections $\nabla_V : V \rightarrow V \otimes_A \Omega$ and $\nabla_W : W \rightarrow W \otimes_A \Omega$ we now construct the connection $\nabla_V \oplus_{\mathcal{R}} \nabla_W : V \otimes_A W \rightarrow V \otimes_A W \otimes_A \Omega$ on the tensor product module $V \otimes_A W$. Since connections are \mathbb{K} -linear maps and not A -linear we actually have to consider their sum on $V \otimes W$; only later we can then consider the tensor product $V \otimes_A W$.

Theorem 4.4. (Sum of connections). *Let (H, \mathcal{R}) be a quasitriangular Hopf algebra and $A \in {}^H\mathcal{A}$, $V, W \in {}^H\mathcal{M}_A$ be quasi-commutative. Let also $(\Omega^\bullet, \wedge, d)$ be a graded quasi-commutative left H -covariant differential calculus over A , $\nabla_V \in \text{Con}_A(V)$ and $\nabla_W \in \text{Con}_A(W)$. Consider the \mathbb{K} -linear map $\nabla_V \hat{\oplus}_{\mathcal{R}} \nabla_W : V \otimes W \rightarrow V \otimes_A W \otimes_A \Omega$ defined by*

$$\nabla_V \hat{\oplus}_{\mathcal{R}} \nabla_W := \tau_{\mathcal{R}23}^{-1} \circ \pi \circ (\nabla_V \otimes_{\mathcal{R}} \text{id}) + \pi \circ (\text{id} \otimes_{\mathcal{R}} \nabla_W) , \quad (4.11)$$

where π denotes the projections $V \otimes_A \Omega \otimes W \rightarrow V \otimes_A \Omega \otimes_A W$ and $V \otimes \Omega \otimes_A W \rightarrow V \otimes_A \Omega \otimes_A W$, and $\tau_{\mathcal{R}23}^{-1}$ is the inverse braiding map acting on the second and third entry of the tensor product $V \otimes_A \Omega \otimes_A W$.

The map $\nabla_V \hat{\oplus}_{\mathcal{R}} \nabla_W$ induces a connection on the quotient module $V \otimes_A W$,

$$\nabla_V \oplus_{\mathcal{R}} \nabla_W : V \otimes_A W \rightarrow V \otimes_A W \otimes_A \Omega , \quad (4.12)$$

defined by, for all $v \in V, w \in W$, $(\nabla_V \oplus_{\mathcal{R}} \nabla_W)(v \otimes_A w) := (\nabla_V \hat{\oplus}_{\mathcal{R}} \nabla_W)(v \otimes w)$, and extended by linearity to all $V \otimes_A W$.

The proof of this theorem relies on the braided Leibniz rule (4.9).

The properties of the $\otimes_{\mathcal{R}}$ -tensor product imply that the sum of connections is compatible with the Hopf algebra action, for all $\xi \in H$, $\xi \blacktriangleright (\nabla_V \oplus_{\mathcal{R}} \nabla_W) = (\xi \blacktriangleright \nabla_V) \oplus_{\mathcal{R}} (\xi \blacktriangleright \nabla_W)$, and that it is associative,

$$(\nabla_V \oplus_{\mathcal{R}} \nabla_W) \oplus_{\mathcal{R}} \nabla_Z = \nabla_V \oplus_{\mathcal{R}} (\nabla_W \oplus_{\mathcal{R}} \nabla_Z) . \quad (4.13)$$

This latter property uses also the braid relation $\tau_{\mathcal{R}2(34)}^{-1} = \tau_{\mathcal{R}34}^{-1} \circ \tau_{\mathcal{R}23}^{-1}$ of Lemma 4.3.

We end this section by mentioning connections on dual modules $V' = \text{Hom}_A(V, A)$ (with V finitely generated and projective). If $V \in {}^H\mathcal{M}_A$ then $V' \in {}^H_A\mathcal{M}$, and a right connection $\nabla_V \in \text{Con}_A(V)$ induces a canonical *left* connection on V' , $\nabla' : V' \rightarrow \Omega \otimes_A V'$. We denote by $\langle v', v \rangle$ the evaluation of v' on v , then (with obvious abuse of notation) ∇' is defined by, for all $v' \in V', v \in V$,

$$\langle \nabla' v', v \rangle = d\langle v', v \rangle - \langle v' \otimes_A \text{id}, \nabla_V v \rangle, \quad (4.14)$$

and satisfies the left Leibniz rule $\nabla'(av') = da \otimes_A v' + a\nabla'(v')$. If we now consider a quasi-commutative A -bimodule $V \in {}^H_A\mathcal{M}_A$ and the Hopf algebra H is triangular we also have a canonical right connection $\nabla_{V'} \in \text{Con}_A(V')$, defined by

$$\langle \nabla_{V'} v', v \rangle = d\langle v', v \rangle - \langle (\bar{R}^\alpha \triangleright v') \otimes_A \text{id}, (\bar{R}_\alpha \blacktriangleright \nabla_V v) \rangle. \quad (4.15)$$

For example if V is the module of vector fields on a noncommutative manifold then V' is that of one-forms, and given $\nabla \in \text{Con}_A(V)$, we can then consider $\nabla_V \oplus_{\mathcal{R}} \nabla_{V'}$, the connection on covariant and contravariant tensor fields $V \otimes_A V'$.

4.3 Twisting sums of connections

The twist quantized sum of connections, $D_{\mathcal{F}}(\nabla_V \oplus_{\mathcal{R}} \nabla_W) : (V \otimes_A W)_\star \rightarrow (V \otimes_A W \otimes_A \Omega)_\star$ is by construction a \mathbb{K} -linear map. As in the case of Theorems 3.4 and 3.5, up to φ -isomorphisms it is a connection.

Theorem 4.5. *Let (H, \mathcal{R}) be a quasitriangular Hopf algebra with twist $\mathcal{F} \in H \otimes H$ and $A \in {}^H\mathcal{A}$, $V, W \in {}^H_A\mathcal{M}_A$ be quasi-commutative. Let further $(\Omega^\bullet, \wedge, d)$ be a graded quasi-commutative left H -covariant differential calculus $\nabla_V \in \text{Con}_A(V)$ and $\nabla_W \in \text{Con}_A(W)$. Then the following diagram commutes*

$$\begin{array}{ccc} V_\star \otimes_{A_\star} W_\star & \xrightarrow{\tilde{D}_{\mathcal{F}}(\nabla_V) \oplus_{\mathcal{R}\mathcal{F}} \tilde{D}_{\mathcal{F}}(\nabla_W)} & V_\star \otimes_{A_\star} W_\star \otimes_{A_\star} \Omega_\star \\ \varphi_{V,W} \downarrow & & \downarrow \varphi_{V,W,\Omega} \\ (V \otimes_A W)_\star & \xrightarrow{D_{\mathcal{F}}(\nabla_V \oplus_{\mathcal{R}} \nabla_W)} & (V \otimes_A W \otimes_A \Omega)_\star \end{array} \quad (4.16)$$

where $\varphi_{V,W,\Omega} = \varphi_{V \otimes_A W, \Omega} \circ (\varphi_{V,W} \otimes_{\mathcal{R}} \text{id}_{\Omega_\star})$ is the diagonal in the commutative square (3.11) (with the substitutions $\otimes_\star \rightarrow \otimes_{A_\star}$).

A special case of the above diagram is when $\mathcal{R} = 1 \otimes 1$ and A is commutative. Then the lower horizontal arrow is the quantization of a usual sum of connections on A -bimodules. The upper horizontal arrow is then the sum of noncommutative connections on quasi-commutative bimodules in ${}^{H^\mathcal{F}}_{A_\star}\mathcal{M}_{A_\star}$, where the Hopf algebra $H^\mathcal{F}$ has triangular \mathcal{R} -matrix $\mathcal{R}^\mathcal{F} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

5 Curvature

5.1 Curvature of connections and of sum of connections

A connection $\nabla : V \rightarrow V \otimes_A \Omega$ on a right A -module V can be extended to a well defined \mathbb{K} -linear map $\nabla : V \otimes_A \Omega^\bullet \rightarrow V \otimes_A \Omega^\bullet$ by setting, for all $v \in V, \theta \in \Omega^\bullet$,

$$\nabla(v \otimes_A \theta) = (\nabla v) \wedge \theta + v \otimes_A d\theta. \quad (5.1)$$

The curvature R_∇ of the connection ∇ is the \mathbb{K} -linear map defined by

$$R_\nabla := \nabla \circ \nabla : V \rightarrow V \otimes_A \Omega^2. \quad (5.2)$$

It is a standard proof to show that it is a right A -linear map, i.e., $R_\nabla \in \text{Hom}_A(V, V \otimes_A \Omega^2)$.

If we have a quasitriangular Hopf algebra (H, \mathcal{R}) , and $A \in {}^H\mathcal{A}$, $V, W \in {}^H\mathcal{M}_A$ are quasi-commutative, and if $(\Omega^\bullet, \wedge, d)$ is a graded quasi-commutative left H -covariant differential calculus, then we can consider the sum of two connections $\nabla_V \in \text{Con}_A(V)$ and $\nabla_W \in \text{Con}_A(W)$. The corresponding curvature $R_{\nabla_V \oplus \mathcal{R} \nabla_W} \in \text{Hom}_A(V \otimes_A W, V \otimes_A W \otimes_A \Omega^2)$ can be shown to satisfy the identity

$$\begin{aligned} R_{\nabla_V \oplus \mathcal{R} \nabla_W} &= \tau_{\mathcal{R}23}^{-1} \circ (R_{\nabla_V} \otimes_{\mathcal{R}} \text{id}_W) + \text{id}_V \otimes_{\mathcal{R}} R_{\nabla_W} \\ &\quad + (\text{id}_{V \otimes_A W} \otimes_{\mathcal{R}} \wedge) \circ \tau_{\mathcal{R}23}^{-1} \circ \left(\nabla_V \otimes_{\mathcal{R}} \nabla_W - (\bar{R}^\alpha \blacktriangleright \nabla_V) \otimes_{\mathcal{R}} (\bar{R}_\alpha \blacktriangleright \nabla_W) \right), \end{aligned} \quad (5.3)$$

where $R_{\nabla_V} \in \text{Hom}_A(V, V \otimes_A \Omega^2)$ and $R_{\nabla_W} \in \text{Hom}_A(W, W \otimes_A \Omega^2)$ are the curvatures of ∇_V and ∇_W , respectively. The second line in (5.3) is a right A -linear map even though the single addends are not.

We remark that in case one of the two connections is H -equivariant, then the second line in (5.3) vanishes, and (5.3) shows that the curvature of the sum of connections is the sum of the curvatures of the initial connections.

5.2 Curvature of twisted connections and twisted curvatures

Let H be a Hopf algebra with twist $\mathcal{F} \in H \otimes H$ and let $A \in {}^H\mathcal{A}$, $V \in {}^H\mathcal{M}_A$, $\nabla \in \text{Con}_A(V)$. We twist these structures to $A_\star \in {}^{H^\mathcal{F}}\mathcal{A}$, $V_\star \in {}^{H^\mathcal{F}}\mathcal{M}_{A_\star}$, $\nabla_\star = \tilde{D}_\mathcal{F}(\nabla) \in \text{Con}_{A_\star}(V_\star)$. The isomorphism $\tilde{D}_\mathcal{F}$ between $\text{Con}_A(V)$ and $\text{Con}_{A_\star}(V_\star)$ can be shown to lift to an isomorphism between extended connections $\nabla : V \otimes_A \Omega^\bullet \longrightarrow V \otimes_A \Omega^\bullet$ and $\nabla_\star : V_\star \otimes_{A_\star} \Omega_\star^\bullet \longrightarrow V_\star \otimes_{A_\star} \Omega_\star^\bullet$, we have

$$\nabla_\star = \varphi_{V, \Omega^\bullet}^{-1} \circ D_\mathcal{F}(\nabla) \circ \varphi_{V, \Omega^\bullet}. \quad (5.4)$$

We can then express the curvature R_{∇_\star} of the quantized connection $\nabla_\star = \tilde{D}_\mathcal{F}(\nabla)$ in terms of the original connection ∇ . We have (use (2.17))

$$R_{\nabla_\star} := \nabla_\star \circ \nabla_\star = \varphi_{V, \Omega^2}^{-1} \circ D_\mathcal{F}(\nabla) \circ \varphi_{V, \Omega} \circ \tilde{D}_\mathcal{F}(\nabla) = \varphi_{V_\star, \Omega^2}^{-1} \circ D_\mathcal{F}(\nabla \circ_\star \nabla) = \tilde{D}_\mathcal{F}(\nabla \circ_\star \nabla). \quad (5.5)$$

Notice that the quantized curvature $\tilde{D}_\mathcal{F}(R_\nabla) = \tilde{D}_\mathcal{F}(\nabla \circ \nabla)$ differs from the curvature of the quantized connection $R_{\tilde{D}_\mathcal{F}(\nabla)} = \tilde{D}_\mathcal{F}(\nabla \circ_\star \nabla)$, hence flat connections are in general not mapped into flat connections. The study of the cohomology of twisted connections that are flat could lead to new cohomology invariants or interesting combinations of undeformed ones.

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